Contents

1. Introduction **3**
- 1.1. Contributions and Publications . . . 4
- 1.2. Linear and Mixed-Integer Linear Programming . . . 5
  - 1.2.1. History and Impact . . . 6
- 1.3. LP Solving Approaches and Duality Theory . . . 7
  - 1.3.1. Duality . . . 8
  - 1.3.2. Interior Point Method . . . 11
  - 1.3.3. Simplex Algorithm . . . 12
  - 1.3.4. SoPlex . . . 15
  - 1.3.5. Choosing the Method . . . 15
- 1.4. MILP Solving Approaches . . . 16
  - 1.4.1. SCIP Optimization Suite . . . 19
  - 1.4.2. Other Solvers . . . 21
- 1.5. Testing Methodology . . . 22

2. PySCIPOpt **25**
- 2.1. Concept . . . 26
  - 2.1.1. Modeling . . . 26
  - 2.1.2. Feature Development . . . 27
- 2.2. Technical Details . . . 27
- 2.3. Performance . . . 28
- 2.4. Licensing and Impact . . . 30

3. Implementational Aspects of the Simplex Algorithm **33**
- 3.1. Stable Summation . . . 33
- 3.2. Row Representation . . . 34
  - 3.2.1. Addition of Cutting Planes . . . 37
  - 3.2.2. Column Generation . . . 38
  - 3.2.3. Different Problem Dimensions . . . 38
- 3.3. Shifting to Generate a Feasible Basis . . . 40

3.4. Long Steps in the Dual Simplex . . . 41
3.4.1. Mathematical Background . . . 41
3.4.2. Dual Pivot . . . 42
3.4.3. Technical Improvements . . . 44
3.4.4. Performance Impact . . . 45
3.5. Pricing Variants . . . 46
3.5.1. Steepest Edge Pricing . . . 46
3.5.2. Devex Pricing . . . 46
3.5.3. Shadow Pricing . . . 47
3.5.4. Parallel Pricing Rules . . . 47
3.5.5. Automatic Pricing Rule Selection . . . 47
3.6. Exploiting Sparsity . . . 48
3.7. Persistent Scaling . . . 51
3.7.1. Scaling Methods . . . 53
3.8. LU Factorization and Update . . . 55
3.9. Iterative Refinement . . . 57
3.10. Decomposition Based Dual Simplex . . . 58
3.11. Performance Variability . . . 59
3.12. Performance Impact of Selected Features . . . 59

4. Impact of Linear Programming in MILP **63**
4.1. Implementational Details . . . 64
4.2. Root and Node LPs . . . 66
4.3. Branching Rules . . . 70
4.4. Node Selection . . . 74
4.5. Cutting Planes . . . 75
4.6. Conflict Analysis . . . 78
4.7. Primal Heuristics . . . 78
4.8. Visualization of MILP Search Trees . . . 79
4.9. Computational Study . . . 83

5. LP Solution Polishing **91**
5.1. Related Work . . . 92
5.2. Description of the Approach . . . 92
5.3. Impact on Numerical Stability . . . 94
5.4. Reduced Costs on the Optimal Facet . . . 95
5.4.1. Reduced Cost Strengthening . . . 96
5.4.2. Cutting Plane Evaluation . . . 97
5.5. Computational Study . . . 97
5.6. Conclusion . . . 99

6. Numerics in Branch & Bound & Cut **101**
6.1. Background on Numerical Analysis . . . 102

6.2. Numerical Analysis for LP and MILP 107
6.2.1. Conditioning of the Simplex Algorithm 110
6.2.2. Condition Number Trend in the Tree 112
6.3. Tailing-off Effect of Cutting Planes 113
6.4. LP Condition Numbers for MILPs 116
6.5. Geometry of the Polyhedron and its Impact on Branching 119
6.6. Exact MILP Solving 119
6.7. Outlook and Future Work 120

7. Conclusion **123**

Bibliography **125**

A. Mittelmann Benchmark Plots **137**

B. Experimental Data and Results **141**